
Math 4550 - Homework # 6 - Normal subgroups and Factor Groups

Part 1 - Computations

1. For the following groups G and subgroups H compute the left and right cosets. Are they equal? Is H a normal subgroup? Draw a picture of how the cosets partition the group G .
 - (a) $G = \mathbb{Z}_{12}$ and $H = \langle \bar{4} \rangle$.
 - (b) $G = D_8$ and $H = \langle r^2 \rangle$.
 - (c) $G = D_8$ and $H = \langle s \rangle$.
2. Let $G = \mathbb{Z}_{12}$ and $H = \langle \bar{4} \rangle$.

H is a normal subgroup of G . You don't have to check this fact.

 - (a) Calculate the elements of G/H . What is the identity element of G/H ?
 - (b) Calculate $(\bar{2} + H) + (\bar{3} + H)$. Find the inverse of $\bar{3} + H$.
 - (c) Find the order of $\bar{1} + H$ in G/H . Find the order of $\bar{2} + H$ in G/H .
 - (d) Is G/H abelian? Is G/H cyclic? If G/H is cyclic, state a generator.
3. Let $G = D_8$ and $H = \langle r^2 \rangle = \{1, r^2\}$.

H is a normal subgroup of G . You don't have to check this fact.

 - (a) Calculate the elements of G/H . What is the identity element of G/H ?
 - (b) Calculate $(rH)(sH)$ and $(srH)(srH)$.
 - (c) Find the inverse of rH . Find the inverse of sH .
 - (d) Find the orders of H , rH , sH , and srH in G/H .
 - (e) Is G/H cyclic? If G/H is cyclic, state a generator.
4. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $H = \langle (\bar{0}, \bar{1}) \rangle$.

H is a normal subgroup of G . You don't have to check this fact.

 - (a) Calculate the elements of G/H . What is the identity element of G/H ?
 - (b) Calculate $[(\bar{1}, \bar{2}) + H] + [(\bar{1}, \bar{1}) + H]$ and $[(\bar{0}, \bar{1}) + H] + [(\bar{2}, \bar{1}) + H]$.
 - (c) Find the inverse of $(\bar{1}, \bar{2}) + H$. Find the inverse of $(\bar{2}, \bar{2}) + H$.
 - (d) Find the orders all the elements in the group.
 - (e) Show that G/H cyclic and list all of its generators.

Part 2 - Proofs

5. Let G be a group and $H \leq G$. Let $a, b \in G$.
- (a) Prove that $aH = bH$ iff $a \in bH$.
 - (b) Prove that $aH = bH$ iff $a = bh$ for some $h \in H$.
 - (c) Show that if $|H|$ is finite, then $|H| = |aH|$.
[Hint: Show that $f : H \rightarrow aH$ where $f(h) = ah$ is one-to-one and onto.]
6. Let G be a group where $|G| = pq$ where p and q are primes and $p \neq q$.
Let H be a subgroup of G with $H \neq G$. Prove that H is cyclic.
7. Let G be a group with identity element e . Suppose that $|G| = n$. Prove that $x^n = e$ for all $x \in G$.
8. Let G be a group and H be a normal subgroup of G .
- (a) Prove that if G is a abelian, then G/H is abelian.
 - (b) Show that $(aH)^{-1} = (a^{-1})H$ for any $a \in G$.
 - (c) Show that $(aH)^k = (a^k)H$ for any $a \in G$ and integer k .
 - (d) Prove that if G is a cyclic, then G/H is cyclic.
9. Let G be a group and H and K be normal subgroups of G . Prove that $H \cap K$ is a normal subgroup of G .
10. Let G_1 and G_2 be groups. Let $\phi : G_1 \rightarrow G_2$ be a homomorphism. Show that the kernel of ϕ is a normal subgroup of G_1 .
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Part 3 - The problem below is OPTIONAL.

11. Let G be a finite group and H be a subgroup of G . Prove that if H is only subgroup of G of size $|H|$, then H is normal in G .
[Hint: Given $g \in G$, consider $\phi_g : H \rightarrow G$ given by $\phi_g(h) = ghg^{-1}$. Use the fact that the image of a homomorphism is a subgroup.]
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